
Problem Solving and Mathematical Reasoning

Areas of Concern for Students

A problem is simply a request for a satisfactory outcome to a given situation. Solving the problem requires a method of organizing the given information and using that information, along with personal knowledge, to obtain an outcome or solution.

Human beings are born problem solvers. Babies, toddlers, and young children learn through problem solving. Although it sounds simple, problem solving is a very difficult skill for many people. Teaching problem solving is a challenge for many teachers. This section of the GED Mathematics Training Institute Manual will focus on problem solving and its relationship to teaching mathematics.

To apply skills in mathematics successfully, students must be good problem solvers. However, many adults in GED programs struggle with problem solving, especially problems with multi-step operations.

Students who struggle with problem solving often fail to:

- Read the problem carefully and pay attention to detail.
- Define the type of answer required and eliminate extraneous information.
- Identify key words that will assist in choosing the correct operation or, in the case of multi-step problems, the correct operations.
- Identify a strategy that will work in solving the problem.
- Use a graphic organizer.
- Set up the problem correctly and remember the order of operations.
- Use mental math and estimation skills.
- Check the answer for reasonableness.
- Use a calculator with care and always double-check the answers.

For most students, the important part of working on a word problem is arriving at an answer. However, students need to learn that problem solving is a process and that when they learn the process they can transfer that knowledge to other types of problems. In the GED classroom, teachers need to spend time working with students through each of the steps listed above. The following information can assist teachers in this process.

Read the Problem Carefully

The first mistake students make when trying to solve a problem is not to read carefully. They think they know what the problem is asking them to do. Unfortunately, they

make the wrong decision. Reading comprehension skills are important in math and often underestimated.

TEACHERS SHOULD

- Spend time reading word problems with students and then deconstructing parts of the problem.
- Help students find important information in the word problem.
- Use a graphic organizer to help students find the main idea of the problem.

Define the Answer Needed

Many students struggle to identify exactly what the question is asking them to do. They fail to pay attention to the details included in the problem and to exclude extraneous information or distractors.

TEACHERS SHOULD

- Provide students with sample problems at a variety of levels of difficulty.
- Have students work in groups and identify the following for each sample problem:
 - What information do they know?
 - What information is not needed?
 - What is the problem asking?
 - How does the answer have to be expressed?
- Have students discuss how they arrived at their decision for each problem.
- Discuss with students any problems they encountered and how they addressed each problem.

Identify Key Words

Many GED students have limited mathematics vocabulary. Students who don't recognize and understand the meaning of terms such as *difference*, *per*, *ratio*, *quotient*, etc., will continue to struggle with word problems. Their limited vocabulary will prevent them from being able to determine what the question is asking them to do.

TEACHERS SHOULD

- Provide students with sample word problems and have them identify key words such as *increase*, *product*, *area*, etc.
- Write the key math terms on charts around the room. Have students add new words as they come across them in different problems.
- Provide students with a math dictionary that they can use to add new words, definitions, and examples.
- Use math words to develop students' vocabulary and then play games such as math jeopardy or bingo.

Introduction to NCTM Method for Problem Solving

Students need a process that they can use consistently when solving problems. George Polya was a great advocate of encouraging the use of problem-solving techniques in learning mathematics. His process involves the four steps listed below. He also outlined numerous strategies for each step, some of which are also included in the following list.

1. Understand the problem

- First, understand the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown or is it insufficient? Redundant? Contradictory?
- Can you draw a figure to introduce a suitable notation?
- Can you write down the various parts of the condition?
- What does the problem measure and what questions must be answered to solve it?

2. Devise a plan

- Second, find the connection between the data and the unknown. If an immediate connection cannot be found, auxiliary problems may need to be considered. Eventually a plan for the solution should be obtained.
- Have you seen the problem before in a slightly different form?
- Do you know a related problem? Do you know a theorem that could be useful?
- Look at the unknown and try to think of a familiar problem having the same or a similar unknown.
- Can you use a problem related to yours that was solved previously? Can you use its result? Can you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Can you restate the problem?
- If you cannot solve the proposed problem, try first to solve some related problem. Can you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Can you solve a part of the problem? Keep only a part of the condition of the problem and drop the other part. How far is the unknown from what needs to be determined? Can you derive something useful from the data in order to determine the unknown? Can you change the unknown so that the new unknown and the new data are nearer to each other?

- Did you use all of the data? Did you use the whole condition? Have you taken into consideration all essential notions involved in the problem?

3. Carry out the plan

- Third, carry out your plan.
- Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove that it is correct?
- If the plan does not seem to be working, start over and try another way.

4. Look back

- Fourth, examine the solution obtained.
- Can you check the result? Can you check the argument? Is the answer reasonable?
- Can you derive the solution differently? Can you see it at a glance? Is there another way of solving the problem that may be easier?
- Can you use the result, or the method, for some other problem?

Adapted from Polya, G. (1954). *How to Solve It*. (2nd ed.). Princeton, NJ: Princeton University Press.

Polya, G. (1980). On Solving Mathematical Problems in High School. In S. Krulik (Ed.), *Problem Solving in School Mathematics* (pp.1–2). Reston, VA: NCTM.

TEACHERS SHOULD

- Review the four-step method with students.
- Provide students with opportunities to try different problems in a low-risk environment so that they can make mistakes and learn from them; share solutions and problems with each other; and develop their own strategies.
- Teach students a variety of strategies for solving different types of problems.

Problem-Solving Strategies

Step 2 in the four-step method requires that students identify a specific strategy to solve a problem. There are many different strategies students can use to solve a problem. In an ideal situation, students should be comfortable with various strategies. Many students coming to the GED classroom have only one or two strategies with which they are comfortable. Provide time for students to explore and discover additional strategies that they can use to become more effective problem solvers. The following are some strategies that should be incorporated into the modeling of good problem solving. Be sure to provide opportunities for students to discuss and explore how best to use these strategies.

WORK BACKWARDS

This strategy requires that the student begin with the end in mind. The student starts with the data presented at the end of the problem and ends with the data at the beginning of the problem. Working backwards is a tricky strategy. It is used when a student isn't given the information that would fit in the beginning of a problem.

MAKE A TABLE, CHART, OR LIST

Another strategy for solving problems is making an organized list. By developing a list, table, or chart, students can count the number of options available or see a pattern to make a decision. Students should be comfortable with both reading and developing tables and charts. The GED Mathematics Test has a variety of graphic displays, including tables and charts. To be successful solving problems that include graphics, students must first know how to construct them. Making a table, chart, or list allows students to put data in an orderly arrangement that enables them to keep track of data they have, find missing data, and clearly identify the data needed to answer a specific problem.

FIND A PATTERN

Students who use this strategy must analyze patterns in data and then make predictions based on the analysis. A pattern is a regular, systematic repetition that may be numerical, visual, or behavioral. When students identify the pattern, they can predict what will come next and what will happen repeatedly in the same way. Finding patterns is an important problem-solving strategy.

DRAW A PICTURE OR MAKE A MODEL

Sometimes it helps if the student can actually see and/or touch the problem. In this case, the student may choose to draw a picture or diagram or even make a model. Objects and pictures can help the student visualize the problem. Although most students have problems with writing equations, equations are an abstract way of modeling a problem. Drawing a picture or making a model can work really well for kinesthetic learners who enjoy hands-on experiences.

GUESS, CHECK, AND REVISE

Although "guess and check" is the most frequently used problem-solving strategy, many students forget to implement the third step in the process—revise. If students spent more time revising, they would have fewer errors. When using this process, it is important that students make a reasonable guess. They should then compute the problem, check the guess that they made, and then revise if necessary. Although this strategy can be tedious if the correct solution is not found soon, students should be encouraged to use this strategy when they don't know another strategy to use to solve a specific problem.

COMPUTE OR SIMPLIFY

Some problems require that the student use specific arithmetic rules. When solving these problems, the student applies the rule or rules needed and calculates the answer. Students must be careful to use the correct order of operations when computing an answer.

USE A FORMULA

Using a formula is an essential strategy for students preparing for the GED Mathematics Test, as well as for solving real-life math problems. Just like using a calculator, students should view formulas as tools for completing math problems. While students do not have to memorize formulas for the GED Mathematics Test (a formulas page is included in the test booklet), they should know basic formulas to solve real-life problems, including distance formulas, perimeter, area, volume, and conversion of temperature from Fahrenheit to Celsius or vice versa. These formulas can help them solve real-life problems, such as how much paint to purchase for a room or the square footage of carpet needed for an apartment or house.

CONSIDER A SIMPLER CASE

Multi-step problems are some of the most difficult for students to solve. Often, students complete only a portion of the problem and thus end up with the wrong answer. Help students avoid these types of errors by teaching them how to consider a simpler case or break down a large problem into mini-problems. Sometimes students can substitute smaller numbers to make it easier to understand. Then they can better see the patterns or relationships among the numbers.

When a problem seems complex or has many parts to it, breaking it down into smaller problems is an excellent strategy. It is much easier to solve these kinds of problems in little steps than to try to solve it all at once.

PROCESS OF ELIMINATION

People use the process of elimination everyday. In math, it is possible to use the process of elimination to find solutions to problems. Sometimes this process is much easier than trying to set up an equation, use a formula, or apply some other problem-solving strategy.

USING MANIPULATIVES

Individual students learn in different ways. Manipulatives allow for the incorporation of different learning styles into the learning process. Students can touch and move objects to make visual representations of mathematical concepts. Manipulatives can be used to represent both numbers and operations on those numbers. Manipulatives also help teachers in assisting students to explore and discover information in new and different ways. Ideally, manipulatives should be available for students to use at any time to help them think, reason, and solve problems.

Graphic Organizers for Problem Solving

Graphic organizers are commonly used in reading and writing. They are also very useful in the mathematics classroom. There are many different types of graphic organizers that contribute both to reading mathematical problems and to addressing problem solving. A graphic organizer is a visual representation of concepts, knowledge, or information that can incorporate both text and pictures. Examples include such things as: Venn diagrams, brainstorming webs, mapping, and flow charts. Graphic organizers allow the learner to visualize undiscovered patterns and relationships.

Examples of graphic organizers from the Access Center include the following:

Hierarchical Graphic Organizer

The following organizer shows different types of polynomials and provides both examples and non-examples:

POLYNOMIALS

MONOMIAL (polynomial of one term)	BINOMIAL (polynomial of two terms)	TRINOMIAL (polynomial of three terms)
5	$5a + 5b$	$5a + 6c + 12d$
x	$10h + 10i$	$x + 2x^2 + 4x^3$
5b	$10 + 12i$	$x^2 + 3x^2 + 6x$ (non-example)
1/5	$7y - 2x$	$3 + 4x + x^2$
10/2	$3x - 4x$ (non-example)	
$5a + 5a$		

Sequence Charts

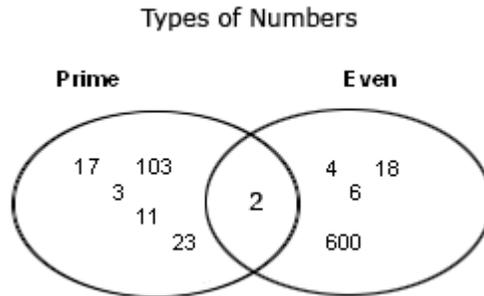
This type of organizer shows the order of a sequence or process. The following is an example of a sequence chart showing Polya's Four-Step Problem Solving Method:

POLYA'S FOUR PROBLEM-SOLVING STEPS

1. Understand the Problem (What is the goal? Draw a representation.)
2. Devise a Plan (Is there a similar problem I can relate to this?)
3. Carry Out the Plan (Carry out plan and check each step.)
4. Look Back (Check the answer for reasonableness.)
<i>Adapted from Polya (1954)</i>

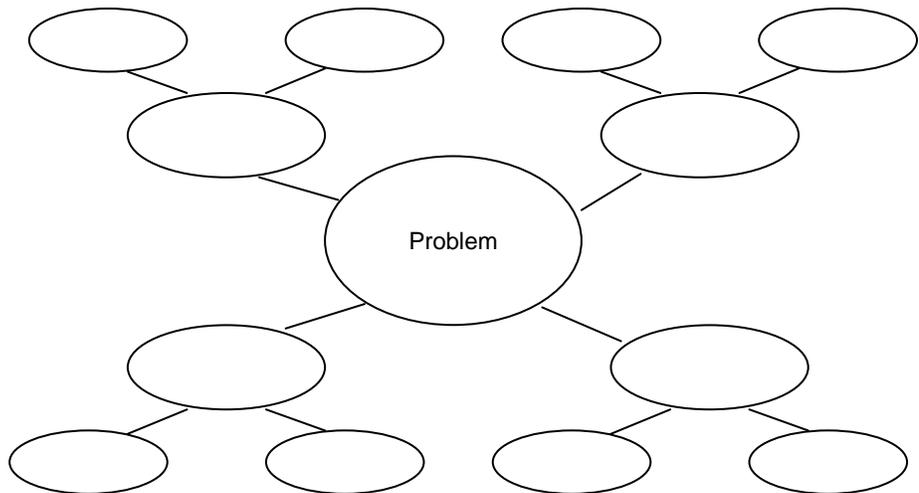
Compare and Contrast or Venn Diagram

This type of graphic compares and contrasts differences and similarities across sets of information. The following compares and contrasts prime and even numbers. It is important that adequate space is provided for students to write.



Brainstorming

This is a useful strategy for developing a highly creative solution to a problem. Brainstorming is a lateral thinking process. It asks that people come up with ideas and thoughts that may at first not seem to connect to the problem. However, brainstorming can be particularly useful when students need to develop new ways of looking at things. By developing a web design, students increase the richness of solutions explored. They can then take these ideas and change and improve them to be useful in solving a problem. The following is an example of a beginning template for a brainstorming web:



Adapted from The Access Center at:
http://www.k8accesscenter.org/training_resources/mathgraphicorganizers.asp.

Hanselman, C.A. (1996). Using Brainstorming Webs in the Mathematics Classroom. *Mathematics Teaching in the Middle School*, 1(9), 766–770. NCTM.



Mind Tools at: <http://www.mindtools.com/brainstm.html>



Some graphic organizers can be used in mathematics to improve the reading comprehension skills necessary to understand the question in word problems. The following graphic organizer was developed by the Texas Center for Adult Literacy and Learning (TCALL) and is part of their adult education toolkit at <http://www-tcall.tamu.edu/toolkit/CONTENTS.HTM>. The original organizer has been modified at the request of adult education practitioners, to incorporate additional elements that may assist students in the problem-solving process.

Before you begin to use the graphic organizer, think about the types of information required. The graphic organizer requires that students:

- Identify the main idea of the problem.
- Determine the question being asked.
- Draw a picture of any physical features described in the problem, such as rectangles, paths, containers, or sets. If several quantities are mentioned, organize the information in a chart or table.
- Make a bulleted list of the important facts in the problem.
- Make a bulleted list of information that is irrelevant and not needed to solve the problem.
- Construct a relationship sentence that does not include numbers. This step requires that students are able to state the relationship of the variables within the problem. What are they trying to solve? What is related to what?
- Set up the equation or number sentence.
- Estimate the answer without computing.
- Compute the answer.
- Write an answer sentence.

The graphic organizer takes students step-by-step through the problem-solving process. As students become more familiar with the problem-solving process, they will not need to use the graphic organizer unless they are confronted with more complex problems.

In the classroom, teachers will need to take students through the organizer step-by-step. This is not an activity that can be completed in one class period. However, it is a

process that will help students focus on details and avoid careless errors in solving problems and therefore is worth the time required.

Teachers Should

- Introduce the entire graphic organizer and explain each part.
- Spend time during each class period focusing on a specific element within the graphic organizer.
- Be sure that students have a clear understanding of each element before moving to the next.
- Provide students with key math words or have them posted on charts around the room to assist students in writing the relationship sentence.
- Have students work in teams to tap into the collective wisdom of the group and encourage team-building skills.

The outline on the following page illustrates how a teacher may wish to introduce the various elements of the graphic organizer. A completed sample graphic organizer is included on page 7-13. A blank graphic organizer, which can be reproduced and used in the classroom, is included on page 7-15.

Suggested Implementation Guidelines for a Word Problem Graphic Organizer

Session 1	<p>Review the purpose of the graphic organizer and briefly discuss each element and how it can assist students in problem solving.</p> <p>Provide students with 3 to 4 simple word problems and have them identify the main idea and the question being asked in each.</p> <p>Do not solve the problems.</p>
Session 2	<p>Use the sample problems from Session 1 and review the main ideas and the question being asked.</p> <p>Have students determine pertinent facts versus irrelevant information.</p> <p>Have students draw a picture/graph/table for the problems (if needed).</p>
Session 3	<p>Review the information from Sessions 1 and 2.</p> <p>As a group, develop a relationship sentence for one problem.</p> <p>Discuss the key words used in building the relationship sentence for one problem.</p> <p>Have students work in teams of 3 or 4 and craft relationship sentences for the remaining problems.</p> <p>Discuss any problems or concerns the students had in developing their sentences.</p> <p>Check for accuracy in their sentences. Each relationship sentence may be a little different from others in the class, but the key elements should be present in each sentence.</p>
Session 4	<p>Review material from Sessions 1–3.</p> <p>Have students write an equation or number sentence based on the relationship sentences. Check for accuracy.</p> <p>Have students estimate the answers to each problem.</p>
Session 5	<p>Review all the information gathered during Sessions 1–4.</p> <p>Have students compute the answers based on their equation or number sentence.</p> <p>Check for accuracy.</p> <p>Have students write an answer sentence for each problem.</p>
Session 6	<p>Review the elements of the graphic organizer.</p> <p>Have students explain in their own words what they need to do for each element.</p> <p>Provide students with one problem.</p> <p>Have them use the graphic organizer to solve the problem.</p> <p>Check for accuracy.</p> <p>Review any areas of concern.</p>

Word Problem Graphic Organizer with Sample Problem

Four boys decided to work together painting houses. For each house they paint, they get \$256.00. Each house will be painted a different color. If the boys work for 4 months and their expenses are \$152.00 per month, how many houses must they paint for each of them to have \$1,000.00?

Main Idea (in your own words) Four boys are painting houses to make an income.	
Question How many houses do they need to paint for each boy to clear \$1,000?	Draw a Picture/Graph/Table
Pertinent Facts <ul style="list-style-type: none">• \$256 per house• 4 months of work• \$152 expenses per month• 4 boys• Want \$1,000 per boy clear	Irrelevant Information Each house is painted a different color.
Relationship Sentence (no numbers) Divide the total amount that the boys want to earn in the given time period and the total amount of expenses for each month by the amount earned per house.	
Equation (number sentence) $4 \times \$1,000 + 4 \times \$152 = \$4,608$ x (number of houses) = $\$4,608/\256 (per month expense cost)	
Estimation (without computing) \$4,000 + estimated expenses divided by \$250 equals more than 16 houses but less than 20.	
Computation $4 \times \$1,000 + 4 \times \$152 = \$4,608$ x (number of houses) = $\$4,608/\256 (per month expense cost) $x = 18$ houses	
Answer Sentence The boys need to paint 18 houses in 4 months in order to each clear \$1,000.	

Word Problem Graphic Organizer

Main Idea (in your own words)	
Question	Draw a Picture/Graph/Table
Pertinent Facts	Irrelevant Information
Relationship Sentence (no numbers)	
Equation (number sentence)	
Estimation (without computing)	
Computation	
Answer Sentence	

More Strategies for Problem Solving

Set Up the Problem Correctly and Remember the Order of Operations

Many students have a basic idea of how to solve a problem, but they fail to set up the problem correctly. One area that causes a lot of concern for students is the order of operations. Students often forget the correct order in which an operation should be calculated, thus ending up with the wrong number. Take a look at the problem below:

$$3 + 2 \times 5 = ?$$

Many students will work from left to right. $3 + 2 = 5$ and then $5 \times 5 = 25$.

The correct order is $2 \times 5 = 10$, $10 + 3 = 13$

Use the mnemonic—**Please Excuse My Dear Aunt Sally**—to help students remember the correct order of operation.

Please complete all operations within **parentheses** first.

Excuse next take care of any **exponents** that may be present.

My Dear complete all **multiplication** and **division**, working from left to right, before moving on to the last operations.

Aunt Sally lastly, perform all **addition** and **subtraction**, working from left to right, and you are done.

Students who are comfortable using the Casio fx-260 Solar Scientific calculator may become complacent about the order of operations. The Casio fx-260 automatically applies the correct order of operations. Students should be aware that they may use a calculator on Part I of the GED Test, but not on Part II. On Part II, they must remember the correct order without prompting.

Use Mental Math and Estimation Skills

Students should be able to look at the answer to a problem and use their mental math or estimation skills to determine if the answer is reasonable. Estimation skills can also be used to eliminate certain answers from the multiple-choice selections. Spend time during each math period working on mental math and estimation skills. This will help students gain confidence in their math ability.

Mental Math

Being able to calculate mentally is an asset in everyday life. There are a number of tricks that students can use to perform mental calculations.

ROUNDING

The first trick to mental math is to use rounding. Round up the number if it ends in 6, 7, 8, or 9 and round down the number if it is 5 or less. Rounding up or down can make numbers easier to manage and allows the mental math to be done quickly.

DOUBLE/HALVE

Another tip for mental math when multiplying is to double one number and halve the other. This can be used when one number is even.

Example

44×5 . Multiplying by 10 is easier than 5. Halve the first number. 44 becomes 22. Now double the second number from 5 to 10. Multiply 22×10 and you get 220.

COMPENSATION METHOD

In this method, students round up to an easier number with which to compute and then subtract that same amount from the answer.

$$198 + 64$$

Change 198 to 200. $200 + 64 = 264$ (2 too many)

Now subtract 2 from the final answer to get the correct answer of 262.

LEFT-TO-RIGHT METHOD

Sometimes it is easier to add from left to right.

$$87 + 35$$

Start with the tens so $80 + 30 = 110$

Now add the ones $7 + 5 = 12$ ($10 + 2$)

Now add them together $110 + 10 = 120 + 2 = 122$

LOOKING FOR COMPATIBLES

In a series of numbers, it is often easier to add up the compatibles such, as $25 + 75 = 100$ or $18 + 82 = 100$.

$$482 + 75 + 218 + 20 + 5$$

$$(482 + 18) + 200 + (25 + 75)$$

$$500 + 200 + 100 = 800$$



Adapted from Numeracy Boost: Background Materials for Adult Learners in Mathematics (2001). Retrieved May 6, 2006, from <http://www.nald.ca/FULLTEXT/numboost/wholenum/numsense/2.html>.

Check the Answer for Reasonableness

Students often fail to check and see if the answer they provide for a problem is reasonable. Although it only takes a few seconds to go back and determine if the answer is in the correct unit of measure (minutes, seconds, hours, pounds, ounces, etc.), students often fail to take the time to re-check their answers. Set aside some time before the end of the class period just for checking work. This will help students get in the habit of doing so. Provide problems with answers that use the wrong units and have students find the errors. Reinforce with students the importance of checking for reasonableness.

Double-Check Calculator Answers

How many times have you used a calculator to balance your checkbook and come out with two different totals? Students don't take the time to do the calculation a second time to verify their answers. This can cause major problems on the test. Teach students the importance of calculating an answer two or even three times to verify that all digits have been entered correctly and the right operations/functions were used. There is a big difference in a total that has been calculated based on multiplication versus division. People make mistakes. Input errors are some of the most common problems with the calculator.

What Is Problem Solving?

The following list from Michael E. Martinez provides a comparison of algorithms versus problem solving, as well as some ideas on the topic of problem solving.

- Algorithms are procedures guaranteed to work every time.
- Problem solving is an interaction between a person's experience and the demands of the task.
 1. There is no formula for problem solving.
 2. What constitutes problem solving varies from person to person.
 3. Mistakes made along the way must be accepted as inextricably linked to the problem-solving process.
 4. The problem solver needs to be aware of:
 - The current activity.
 - The overall goal of the activity.
 - The strategies used to attain that goal.
 - The effectiveness of those strategies.
 5. Maintaining flexibility is essential—getting off course is expected.

6. By its very nature, problem solving involves error and uncertainty.
7. The problem solver needs to be willing to accept temporary uncertainty.
8. Anxiety is a spoiler of the process.
9. Errors are part of the process—therefore—instructors who are themselves unsure, or who are unwilling to admit their own uncertainty, will not permit the full exploration of a problem.
10. Fixed knowledge and algorithms are easier to teach, learn, and test.
11. Educators must accept errors, uncertainty, and indirect paths toward solutions.

From Martinez, Michael E. (April 1998). What Is Problem Solving? *Phi Delta Kappan*.